Range of multifractality for random walks on random fractals

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We study the range of multifractality for the probability density $P(r,t)$ of random walks on linear random fractals, for a given distance r and time t. Analytical study of the moments $\langle P^q(r,t)\rangle$ shows that multifractality exists only when $1<qr^{d_w}/t$ and $qr/t<1$, with $d_w=2d_f$, where d_f is the fractal dimension of the linear fractal. The results can be extended to more general random fractals and are consistent with recent numerical data for the form of $\langle P(r,t)\rangle$.

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Several recent studies have demonstrated that in a wide range of dynamical processes in random media, the usual scaling laws fail to exist [1–9]. Rather, the quantities characterizing these processes have anomalously broad distributions. Their moments cannot be described by a single exponent but an infinite hierarchy of exponents is needed to characterize them. This phenomenon is called multifractality. Some examples are the growth probabilities in diffusion-limited aggregates (DLA) [1–3], the voltage drops in percolation clusters [4], the probability density of random walks on fractals [7], the tracer concentration in stratified media with random fields [8], and the amplitudes of vibrational excitations (fractons) in percolation systems [9].

An exactly solvable model is a random walk on a linear chain, generated by a simple random walk in d dimensions (RW on RW chain). Although in this model the chain can intersect itself in the space, the walker is constrained to follow the topologically one-dimensional path which has been created sequentially by a previous random walk. The basic quantity describing the spatial extension of the random walk on the chain is the probability density $P(r,t)$ that the walker is at a given distance r at time t from its starting point $r=0$ at $t=0$. As we have shown recently, the fluctuations of $P(r,t)$ display multifractal features [7] characterized by a nonstandard behavior of the moments $\langle P^q(r,t)\rangle$ $(q>0)$ and by an anomalously broad distribution $N(\ln P)$ of the values $P\equiv P(r,t)$ between $|\ln P|$ and $|\ln P|+d|\ln P|$.

In this article we show that this multifractal behavior does not occur for all values $q>0$. Rather, we find that the conditions on q, which are necessary for the multifractality to appear, are

$$q_{\min}\equiv\frac{t}{r^{d_w}}<q<\frac{t}{r}\equiv q_{\max}\ . \tag{1}$$

Here d_w is the anomalous diffusion exponent describing the behavior of the mean-square displacement of the random walk, $R^2\equiv\langle r^2\rangle\sim t^{2/d_w}$. For linear fractals, $d_w=2d_f$, where d_f is the fractal dimension of the fractal chain. In the case of RW chains, $d_f=2$ and $d_w=4$. Outside the range (1), the multifractality breaks down and standard scaling behavior takes place.

To show this, we follow Ref. [7] and start with the definition of the configurational average,

$$\langle P^q(r,t)\rangle\equiv\frac{1}{N_r}\sum_{i=1}^{N_r}P_i^q(r,t)\ , \tag{2}$$

where the sum is over all N_r sites i of the chain located at a distance r from the origin, and $P_i(r,t)$ denotes the probability to be at site i. Note that N_r includes many configurations with a very large number of sites.

After time t, the random walk can be at many different sites i at distance r from the origin, and their corresponding probabilities $P_i(r,t)$ may obtain very different values according to their *chemical* distance to the origin. The crucial point is that *along* the chain the probability to be at the chemical distance l is *independent* of the chain configuration in space and given simply by a Gaussian,

$$P(l,t)\sim t^{-1/2}\exp(-al^2/t) \tag{3}$$

for $l<t$, and $P(l,t)=0$ for $l>t$. Thus the sum in (2) can be written as a sum over sites having the same l values,

$$\langle P^q(r,t)\rangle = \sum_l \frac{N_l}{N_r} P^q(l,t) \ , \tag{4}$$

where N_l denotes the number of those sites which are at chemical distance l from the origin. Since $P(l,t)$ does not depend on the chain configuration, its average moments are simply $P^q(l,t)$.

By definition, $N_l/N_r \equiv \phi(l|r)$ is the probability that two sites separated a distance r are at chemical distance l. Transforming the sum (4) into an integral over l yields

$$\langle P^q(r,t)\rangle = \int_r^t dl\ \phi(l|r) P^q(l,t) \ . \tag{5}$$

The lower integration limit $l=r$ is due to the fact that $\phi(l|r)=0$ when $l<r$ (see also [9]), while the upper limit comes from the condition that $P(l,t)=0$ when $l>t$.

For the RW chain, the structural function ϕ is proportional to a Gaussian,

$$\phi(l|r) \sim (1/l)(r^2/l)^{d/2-1} \exp(-ar^2/l), \quad l>r \ , \tag{6}$$

which satisfies the normalization condition $\sum_l N_l/N_r = 1$, which follows from (4) with $q=0$. Note that $\phi(l|r)$ in (6) is normalized only for $d>2$.

We consider here the large-time case $t>r^2$, where the upper integration limit $l=t$ in (5) corresponds to values of l well beyond the position of the maximum of $\phi(l|r)$ at $l\sim r^2$. According to (6), $\phi(l|r)$ increases exponentially as $\exp(-r^2/l)$ for $r<l<r^2$ and decreases as $(1/l)^{d/2}$ for $l>r^2$.

We proceed by evaluating the integral (5) by using the method of steepest descent. The saddle occurs at

$$l = l_0 = (r^2 t/q)^{1/3} \ , \tag{7}$$

which yields

$$\langle P^q(r,t)\rangle \sim t^{-q/2} \exp[-\text{const} \times q^{\tau} (r/R)^u] \ , \tag{8a}$$

with $\tau = \frac{1}{3}$ and $u = \frac{4}{3}$. The nonlinear q dependence of the exponent in (8a) indicates multifractal behavior of the moments in r space as discussed in [7]; see also [1,5,8,9]. The result (8a) holds only when $r<l_0<r^2$. This is because the saddle point must be both in the integration range and where the exponential behavior dominates the algebraic term in (6).

For $l_0>r^2$, $qr^4/t<1$ and the exponent term in $P^q(l,t)$ does not have much of an effect on the position of the maximum. The integral in (5) is dominated by values of $l\sim r^2$ near the maximum of $\phi(l|r)$ and we find

$$\langle P^q(r,t)\rangle \sim t^{-q/2} \phi(r^2|r) \exp[-\text{const} \times (qr^4/t)] \ , \quad l_0 > r^2 \ . \tag{8b}$$

Equation (8b) indicates that multifractality breaks down in the regime $l_0>r^2$, i.e., for $qr^4/t<1$, the moments scale in the usual linear way with q.

For $l_0<r$, the integrand is peaked at $l\sim r$ and we obtain

$$\langle P^q(r,t)\rangle \sim t^{-q/2} \phi(r|r) \exp[-\text{const} \times (qr^2/t)] \ , \quad l_0 < r \ , \tag{8c}$$

which implies standard behavior for the moments when $qr/t>1$. Note that for $t<r^2$, $q_{\min} = r^2/t^2$ and for $q<q_{\min}$, $\langle P^q(r,t)\rangle$ has a Gaussian form similar to Eq. (8c).

According to (8), the exponential behavior of the moments changes drastically with q. For $q<t/r^4 \equiv q_{\min}$ we have $\ln\langle P^q(r,t)\rangle \sim -qr^4/t$; in the multifractal regime $q_{\min}<q<t/r \equiv q_{\max}$ we have $\ln\langle P^q(r,t)\rangle \sim -q^{1/3} r^{4/3}/t^{1/3}$; and for $q>q_{\max}$ we have $\ln\langle P^q(r,t)\rangle \sim -qr^2/t$, leading to the range of multifractality given in Eq. (1). We note that analogous crossovers were found for fractons in percolation clusters [9], but there the situation is considerably more complicated.

The above rigorous arguments can be extended to RW's on general linear fractals, where similar considerations can be applied. We assume that $\phi(l|r)$ is a function of the scaling variable r/l^{ν}, i.e.,

$$\phi(l|r) \sim (1/l)(r/l^{\nu})^g \exp[-c(r/l^{\nu})^{\delta}] \ , \tag{9}$$

where $\nu = 1/d_f$ and $\delta = 1/(1-\nu)$ and g is the exponent characterizing the probability of the fractal chain to close a loop [10]. The scaling form (9) holds, e.g., for self-avoiding random walks, where $\nu = 3/(d+2)$, $g = (\gamma-1)/\nu + d - 1/\nu$, and $\gamma = \frac{4}{3}$ for $d=2$ and $\gamma = \frac{7}{6}$ for $d=3$ [10]. Above the critical dimension, $d \geq d_c = 4$, $\gamma = 1$, $\nu = \frac{1}{2}$ and (9) reduces to (6).

Since (3) is exact for random walks on all linear fractals, we can proceed as above for the RW chain. Using (5) and (9), we find the saddle point at

$$l_0 \sim (r^{\delta} t/q)^{1/(\delta+1)} \tag{10}$$

and the multifractal result (8a) follows for the range $r<l_0<r^{1/\nu}$, where now

$$\tau = \frac{1}{d_w - 1} = \frac{\nu}{2-\nu} \tag{11}$$

and $u = d_w/(d_w-1)$.

In this case, our result (8b) can be written as

$$\langle P^q(r,t)\rangle \sim t^{-q/2} \phi(r^{1/\nu}|r) \exp\left[-\text{const} \times \frac{qr^{d_w}}{t}\right] , \quad qr^{d_w}/t \ll 1 \ , \tag{12}$$

while (8c) is changed to

$$\langle P^q(r,t)\rangle \sim t^{-q/2} \phi(r|r) \exp\left[-\text{const} \times \frac{qr^{1/\nu}}{t}\right] , \quad qr/t \gg 1 \ . \tag{13}$$

Equations (12) and (13) reduce to (8b) and (8c) for the RW chain where $\nu = \frac{1}{2}$ and $d_w = 4$. Equation (12) also suggests that for the first moment $q=1$,

$$\langle P(r,t)\rangle \sim t^{-1/2} \exp\left[-\text{const} \times \frac{r^{d_w}}{t}\right], \quad r \ll t^{1/d_w} \ . \tag{14}$$

The above results are rigorous for random walks on linear fractals. For the more general case, random walks on random fractals, one can apply an analogous approach

based on scaling arguments and numerical simulations. For this case, numerical data for percolation clusters at criticality suggest that [11]

$$\phi(l|r)\sim\frac{1}{l}\left[\frac{r}{l^{\tilde{\nu}}}\right]^{\tilde{g}}\exp[-c(r/l^{\tilde{\nu}})]^{1/(1-\tilde{\nu})} \tag{15}$$

and [7,12]

$$\langle P(l,t)\rangle\sim\frac{1}{t^{d_s/2}}\exp[-(l/t^{1/d_w^l})^{\delta_l}]\ ,$$

$$\delta_l=d_w^l/(d_w^l-1)\ . \tag{16}$$

Here d_l is the chemical dimension of the fractal, defined by the scaling of the mass with the chemical distance, $M\sim l^{d_l}$, and $\tilde{\nu}=d_l/d_f$ substitutes ν. The exponent d_w^l is the diffusion exponent in the chemical distance metric, i.e., $\langle l\rangle\sim t^{1/d_w^l}$. The probability density $P(l,t)$ for general fractals is not the same for each site at chemical distance l as for the case of linear fractals. Thus for general fractals Eq. (5) should be replaced by

$$\langle P^q(r,t)\rangle=\int_r^t dl\,\phi(l|r)\langle P^q(l,t)\rangle\ . \tag{17}$$

However, as shown in [7], the fluctuations of $P(l,t)$ are very narrow and therefore we expect that $\langle P^q(l,t)\rangle$ can be replaced by $\langle P(l,t)\rangle^q$. To proceed, we calculate $\langle P^q(r,t)\rangle$ using this assumption and Eqs. (16) and (17). For the steepest descent to be valid, the saddle point

$$l_0=(r^{1/(1-\tilde{\nu})}t^{\delta_l/d_w^l}/q)^{1/[\delta_l+\tilde{\nu}/(1-\tilde{\nu})]}$$

should be in the range $r<l_0<r^{1/\tilde{\nu}}$. This yields the range of moments for which multifractality is expected to occur,

$$q_{\min}=\left[\frac{t}{r^{d_w}}\right]^{1/(d_w^l-1)}<q<\left[\frac{t}{r}\right]^{1/(d_w^l-1)}\equiv q_{\max}\ , \tag{18}$$

which is a generalization of (1). In this range the multifractal result, Eq. (8a) is valid with the prefactor $t^{-q/2}$ replaced by $t^{-d_s q/2}$, $d_s=2d_f/d_w$,

$$\tau=\frac{d_w^l-1}{d_w-1}\ , \tag{19}$$

and $u=d_w/(d_w-1)$. For values of q outside the range given by Eq. (18), the form of $\langle P^q(r,t)\rangle$ will depend on the form of $\langle P(l,t)\rangle$. A reasonable assumption, supported by numerical simulations on regular fractals [13], is that $\langle P(l,t)\rangle$ for $t\gg l^{d_w^l}$ has the form $\langle P(l,t)\rangle\sim t^{-d_s/2}\exp[-l^{d_w^l}/t]$. This yields that for $q^{d_w^l-1}r^{d_w}/t\ll 1$ (or $q\ll q_{\min}$), $\langle P^q(r,t)\rangle$ has the form of (12) with the prefactor $t^{-q/2}$ replaced by $t^{-d_s q/2}$. Thus one obtains for $q=1$

$$\langle P(r,t)\rangle\sim t^{-d_s/2}\exp\left[-\text{const}\times\frac{r^{d_w}}{t}\right],\quad r\ll t^{1/d_w} \tag{20}$$

similar to Eq. (14). We note that Eq. (20) has also been obtained by several recent numerical simulations for random walk on the Sierpinski gasket [13].

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